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THE NON-LINEAR THEORY OF THE PURE BENDING OF PRISMATIC ELASTIC SOLIDS†

A. A. ZELENINA and L. M. ZUBOV

Rostov-on-Don

e-mail: zubov@ms.math.rsu.ru

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The problem of the bending of a prismatic elastic solid by finite torques under large deformation conditions is considered. Using the semi-inverse method, the initial three-dimensional boundary-value problem of the non-linear theory of elasticity is reduced to a two-dimensional non-linear boundary-value problem for a region in the form of the cross-section of the beam. Two formulations of the problem are given in the cross-section: in terms of the displacements and of the stresses. Stress functions are introduced and a variational formulation of the two-dimensional problem is obtained, based on the supplementary energy principle. An approximate solution of the problem of the strong bending of a beam of rectangular cross-section is found for a semi-linear material and for a Bartenev–Khazanovich material using the Ritz method. © 2000 Elsevier Science Ltd. All rights reserved.

In the linear theory of elasticity the problem of the deformation of a prismatic solid by end loads is called the Saint-Venant problem. Hence, the problem of strong pure bending is a non-linear form of one of the Saint-Venant problems. The solution of the other non-linear Saint-Venant problem – the torsion problem – was described previously [1, 2]. The plane non-linear problem of the pure bending of an elastic strip was considered in [3].

1 REDUCTION OF THE PROBLEM OF PURE BENDING TO A TWO-DIMENSIONAL NON-LINEAR BOUNDARY-VALUE PROBLEM

Consider an elastic solid, which, in the reference configuration, has the form of a prismatic rod. The Cartesian coordinates x_1 and x_2 will be read off in the plane of the cross-section of the prism and the x_3 coordinate will be read off along the beam axis. The coordinate unit vectors of the beam will be denoted by $\mathbf{i}_k$ ($k=1, 2, 3$). We will denote by X_n ($n=1, 2, 3$) the Cartesian coordinates of points of the deformed solid, read off along the same directions and from the same origin. We will assume that this elastic solid undergoes a finite deformation, described by the relations

$$X_1 = \alpha(x_1, x_2), \quad X_2 = \rho(x_1, x_2)\cos\beta x_3, \quad X_3 = \rho(x_1, x_2)\sin\beta x_3 \tag{1.1}$$

$$\beta = \text{const}$$

It can be seen that for a deformation of the form (1.1) each material straight line, parallel in this frame of reference to the beam axis, after deformation is converted into an arc of a circle, situated in a plane parallel to the x_2x_3 plane. Any cross-section x_3 = const after deformation remains plane, rotated around the $\mathbf{i}_1$ axis by an angle βx_3 and undergoing a certain deformation in its plane, characterized by the functions $\alpha(x_1, x_2)$, $\rho(x_1, x_2)$. Hence, the coordinate transformation (1.1) converts the right prism into a sector of a circular ring, which corresponds to the bending of the rod.

The deformation gradient, calculated using (1.1), has the form

$$\mathbf{C} = \frac{\partial X_n}{\partial x_k}\mathbf{i}_k\mathbf{i}_n = \frac{\partial \alpha}{\partial x_1}\mathbf{i}_1\mathbf{i}_1 + \frac{\partial \rho}{\partial x_1}\mathbf{i}_1\mathbf{e}_2 + \frac{\partial \alpha}{\partial x_2}\mathbf{i}_2\mathbf{i}_1 + \frac{\partial \rho}{\partial x_2}\mathbf{i}_2\mathbf{e}_2 + \beta\rho\mathbf{i}_3\mathbf{e}_3 \tag{1.2}$$

$$\mathbf{e}_2 = \mathbf{i}_2\cos\beta x_3 + \mathbf{i}_3\sin\beta x_3, \quad \mathbf{e}_3 = -\mathbf{i}_2\sin\beta x_3 + \mathbf{i}_3\cos\beta x_3 \tag{1.3}$$

By (1.3) the vector triple $\mathbf{i}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ forms an orthonormalized basis, and the vectors $\mathbf{i}_1$ and $\mathbf{e}_2$ lie in the cross-section plane of the curved rod. Using (1.2) we can determine the measure of the Cauchy deformations.

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$$\mathbf{G}=\mathbf{C}\cdot\mathbf{C}^T=\left[\left(\frac{\partial\alpha}{\partial x_1}\right)^2+\left(\frac{\partial\rho}{\partial x_1}\right)^2\right]\mathbf{i}_1\mathbf{i}_1+\left[\left(\frac{\partial\alpha}{\partial x_2}\right)^2+\left(\frac{\partial\rho}{\partial x_2}\right)^2\right]\mathbf{i}_2\mathbf{i}_2+$$

$$+\left(\frac{\partial\alpha}{\partial x_1}\frac{\partial\alpha}{\partial x_2}+\frac{\partial\rho}{\partial x_1}\frac{\partial\rho}{\partial x_2}\right)(\mathbf{i}_1\mathbf{i}_2+\mathbf{i}_2\mathbf{i}_1)+\beta^2\rho^2\mathbf{i}_3\mathbf{i}_3 \qquad (1.4)$$

Expression (1.4) shows that the tensor **G** is independent of the x_3 coordinate.

The system of equations of elastostatics of an isotropic uniform solid when there are no mass forces consists of the equilibrium equations for the Piola stress tensors **D**

$$\mathbf{i}_k\cdot\frac{\partial\mathbf{D}}{\partial x_k}=0 \qquad (1.5)$$

and the constitutive relations

$$\mathbf{D}=\frac{dW}{d\mathbf{C}}=a_1(I_1,I_2,I_3)\mathbf{C}+a_2(I_1,I_2,I_3)\mathbf{G}\cdot\mathbf{C}+a_3(I_1,I_2,I_3)\mathbf{C}^{-T}$$

$$I_1=\operatorname{tr}\mathbf{G},\quad I_2=\tfrac{1}{2}(\operatorname{tr}^2\mathbf{G}-\operatorname{tr}\mathbf{G}^2),\quad I_3=\det\mathbf{G} \qquad (1.6)$$

Here W is the specific potential energy of the deformation, I_1, I_2 and I_3 are the principal invariants of the Cauchy deformation measure, and a_1, a_2 and a_3 are certain functions of these invariants.

It follows from (1.2), (1.4) and (1.6) that the Piola stress tensor in this problem has the following representation

$$\mathbf{D}=D_{11}\mathbf{i}_1\mathbf{i}_1+D_{12}\mathbf{i}_1\mathbf{e}_2+D_{21}\mathbf{i}_2\mathbf{i}_1+D_{22}\mathbf{i}_2\mathbf{e}_2+D_{33}\mathbf{i}_3\mathbf{e}_3 \qquad (1.7)$$

where the quantities D_{sk} are independent of the x_3 coordinate. Taking (1.7) into account the equilibrium equations (1.5) can be reduced to the form.

$$\frac{\partial D_{11}}{\partial x_1}+\frac{\partial D_{21}}{\partial x_2}=0,\quad \frac{\partial D_{12}}{\partial x_1}+\frac{\partial D_{22}}{\partial x_2}=\beta D_{33} \qquad (1.8)$$

The boundary conditions $\mathbf{n}\cdot\mathbf{D}=0$ on the side surface of the beam, which is assumed to be load-free, can be written, using (1.7), in the following form

$$n_1D_{11}+n_2D_{21}=0,\quad n_1D_{12}+n_2D_{22}=0$$

$$\mathbf{n}=n_1\mathbf{i}_1+n_2\mathbf{i}_2 \qquad (1.9)$$

Here **n** is the unit vector of the normal to the side surface of the prism.

We will now determine the principal vector and the principal moment of the forces acting in an arbitrary cross-section of the bent beam. We integrate the equilibrium equations (1.5) over part of the volume of the beam, bounded, in the system of reference, by the side surface and the two sections $x_3=a$ and $x_3=b$, where a and b are arbitrary numbers. Using relations (1.7) and (1.9) and applying the divergence theorem, we obtain the equation

$$[(\sin\beta a-\sin\beta b)\mathbf{i}_2+(\cos\beta b-\cos\beta a)\mathbf{i}_3]\iint_\sigma D_{33}d\sigma=0 \qquad (1.10)$$

where σ is the plane region occupied by the cross-section of the beam in the frame of reference employed. If follows from (1.10) that the integral on the left-hand side is equal to zero. This indicates that the principal force vector, acting in any cross-section of the beam for a deformation of the form (1.1). is equal to zero. Consequently, the principal moment of the forces in the cross-section is independent of the point of reduction. Since the vector $\mathbf{e}_3$ is directed along the normal to the plane of cross-section of the prism in the deformed state, it follows from (1.7) that only normal stresses act in the section. For this reason the vector of the principal moment of these stresses has no component along the $\mathbf{e}_3$ axis, i.e. the torque is equal to zero. The vector of the bending moment can be expressed by the formula

$$\mathbf{M} = -\iint_\sigma \mathbf{i}_3 \cdot \mathbf{D} \times (\alpha \mathbf{i}_1 + \rho \mathbf{e}_2) d\sigma = \mathbf{i}_1 \iint_\sigma D_{33} \rho d\sigma - \mathbf{e}_2 \iint_\sigma D_{33} \alpha d\sigma \quad (1.11)$$

Expression (1.11) shows that the bending moments, with respect to orthogonal axes situated in the plane of the deformed cross – section, are the same in all cross-sections of the beam.

Hence, we have shown that the realization of deformation (1.1) in a uniform isotropic prismatic beam requires only a bending moment to be applied to the beam ends. The plane of action of the resulting pair in general is not parallel to the bending plane x_2x_3.

The unknown functions $\alpha(x_1, x_2)$ and $\rho(x_1, x_2)$ are found by solving two-dimensional non-linear boundary-value problem (1.8), (1.9) for the region σ. We will assume that the quantities D_{sk} in (1.8) and (1.9) can be expressed in terms of the functions ρ and α using relations (1.2), (1.4) and (1.6). If α and ρ are the solution of this boundary-value problem, then, as can easily be shown, the functions $\alpha + h$ and ρ, where h is an arbitrary constant, will also satisfy Eqs (1.8) and boundary conditions (1.9). This non-uniqueness of the solution can be eliminated if the function α is subject to the additional condition

$$\iint_\sigma \alpha d\sigma = 0 \quad (1.12)$$

which eliminates the possibility of a free translational displacement of the elastic solid in the direction of the x_1 axis. Taking limitation (1.12) into account we would expect the solution of problem (1.8), (1.9) to be unique. In fact, the non-uniqueness of the solution would imply the existence of forms of loss of stability of the bent rod for which the deformation is the same in all cross-sections of the beam. If this type of equilibrium bifurcation is possible, it would occur for extremely large values of the parameter β.

We will now assume that the cross-section of the beam has a single axis of symmetry, which coincides with the x_2 axis. In this case, physical consideration suggests that the distribution of the normal stresses D_{33} over the cross-section of a beam bent in the X_2X_3 plane will be symmetrical about the x_2 axis, and the moment of these stresses about the x_2 axis will be equal to zero. This assertion can be rigorously proved starting from the uniqueness of the solution of two-dimensional boundary-value problem (1.8), (1.9), (1.12).

It can be shown by a direct check that in the case of a symmetrical cross-section, boundary-value problem (1.8), (1.9), (1.12) for an isotropic uniform material is invariant under the following replacement of the independent variables and unknown functions

$$x_1' = -x_1, \quad x_2' = x_2, \quad \alpha' = -\alpha, \quad \rho' = \rho \quad (1.13)$$

Suppose $\alpha = f(x_1, x_2)$ and $\rho = g(x_1, x_2)$ are the solution of boundary-value problem (1.8), (1.9), (1.12). By virtue of (1.13) the functions $\alpha = -f(-x_1, x_2)$ and $\rho = g(-x_1, x_2)$ satisfy the same boundary-value problem. From the uniqueness of the solution we obtain

$$f(-x_1, x_2) = -f(x_1, x_2), \quad g(-x_1, x_2) = g(x_1, x_2) \quad (1.14)$$

The following property of the solutions of boundary-value problem (1.8), (1.9), (1.12) for the region σ, symmetrical about the x_2 axis, follows from (1.2), (1.4), (1.6) and (1.14): the functions α, D_{12} and D_{21} are odd, while ρ, D_{11} and D_{33} are even with respect to the variable x_1. This property, taking relation (1.11) into account, implies the equality $\mathbf{M} \cdot \mathbf{e}_2 = 0$. Hence, if the cross-section of the beam has an axis of symmetry in the bending plane, the system of forces acting in any cross-section of the deformed beam is statically equivalent to the bending moment acting in the bending plane.

After solving boundary-value problem (1.8), (1.9), (1.12), the value of the moment $M_1 = \mathbf{M} \cdot \mathbf{i}$ is calculated from (1.11) and becomes a certain known function of the parameter β. Inversion of the function $M_1(\beta)$ enables us to determine the curvature of the axis of the bent rod for a specified value of the bending moment.

Thus, assumptions (1.1) on the nature of the deformation of the prismatic solid enable one, by solving a two-dimensional boundary-value problem, to satisfy the equilibrium equations inside the solid and the boundary conditions on the side surface. The boundary conditions at the cylinder ends are satisfied in the Saint-Venant sense, i.e. in the integral sense of the static equivalence of the stresses to the specified bending moment. The initial three-dimensional boundary-value problem for a cylindrical solid is thereby reduced to a two-dimensional problem on the cross-section of the cylinder.

The possibility of replacing the exact boundary conditions at the beam ends by integral relations is based on the Saint-Venant principle. The use of the Saint-Venant principle in non-linear problems of elasticity is related to the singularities [4], which appear when the principal vector of the forces acting at the beam end are non-zero. In the pure bending problem considered here there are no such singularities, since the principal vector of the end load is equal to zero. This enables us, for this non-linear problem, to assume that the Saint-Venant principle hold in the sense that the stressed states of the beam, caused by the action of two different systems of end loads, having zero principal vector and the same principal moment, will be different solely in the immediate vicinity of the end.

2. FORMULATION OF THE TWO-DIMENSIONAL PROBLEM IN STRESSES

We will covert the boundary-value problem described above on a cross-section of a cylinder by eliminating the function α and ρ and taking the unknown components of the Piola stress tensor (1.7) as fundamental. As is well known [5], the deformation gradient satisfies the following compatibility equations

$$\mathbf{i}_k \times \partial \mathbf{C}/\partial x_k = 0 \tag{2.1}$$

By (1.2) in the problem of pure bending the tensor **C** has the representation

$$\mathbf{C} = C_{11}\mathbf{i}_1\mathbf{i}_1 + C_{12}\mathbf{i}_1\mathbf{e}_2 + C_{21}\mathbf{i}_2\mathbf{i}_1 + C_{22}\mathbf{i}_2\mathbf{e}_2 + C_{33}\mathbf{i}_3\mathbf{e}_3 \tag{2.2}$$

where C_{sk} is independent of x_3. Substituting (2.2) into (2.1) and taking (1.3) into account we obtain a system of compatibility equations in the theory of pure bending

$$\frac{\partial C_{21}}{\partial x_1} = \frac{\partial C_{11}}{\partial x_2}, \quad \frac{\partial C_{33}}{\partial x_1} = \beta C_{12}, \quad \frac{\partial C_{33}}{\partial x_2} = \beta C_{22} \tag{2.3}$$

In order to write the compatibility equations in terms of the stresses, we need to express the deformation gradient **C** in terms of the tensor **D**. This can be done by the method described previously in [6]. We will introduce the tensors **U** and **A**, which form a polar expansion of the deformation gradient

$$\mathbf{C} = \mathbf{U} \cdot \mathbf{A} \tag{2.4}$$

Here **U** is a positive-definite quadratic root of the measure of Cauchy deformation and **A** is a strictly orthogonal tensor, which accompanies deformation and the so-called rotation tensor. The tensor **S**, defined by the relation [6]

$$\mathbf{S} = \mathbf{D} \cdot \mathbf{A}^T \tag{2.5}$$

in the case of an isotropic material is symmetrical and is an isotropic function of the tensor **U**

$$\mathbf{S} = dW/d\mathbf{U} = \tau(\mathbf{U})$$

Under conditions, indicated in [6], the function τ is uniquely invertible, i.e. the following relation exists

$$\mathbf{U} = \eta(\mathbf{S}) \tag{2.6}$$

The problem of constructing the relation **C**(**D**) then reduces to representing the rotation tensor in terms of the Piola tensor **A** = **A**(**D**). In fact, from (2.4) – (2.6) we have

$$\mathbf{C} = \eta[\mathbf{D} \cdot \mathbf{A}^T(\mathbf{D})] \cdot \mathbf{A}(\mathbf{D}) \tag{2.7}$$

It follows from relations (1.3) and (2.2) that in the bending problem considered here the rotation tensor can be represented as follows:

$$\begin{aligned} &\mathbf{A}(x_1, x_2, x_3) = \mathbf{H}(x_1, x_2) \cdot \mathbf{Q}(x_3) \\ &\mathbf{H}(x_1, x_2) = (\mathbf{E} - \mathbf{i}_3\mathbf{i}_3)\cos\omega(x_1, x_2) + \mathbf{i}_3\mathbf{i}_3 - \mathbf{i}_3 \times \mathbf{E}\sin\omega(x_1, x_2) \\ &\mathbf{Q}(x_3) = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_1\mathbf{e}_2 + \mathbf{i}_3\mathbf{e}_3 = (\mathbf{E} - \mathbf{i}_1\mathbf{i}_1)\cos\beta x_3 + \mathbf{i}_1\mathbf{i}_1 - \mathbf{i}_1 \times \mathbf{E}\sin\beta x_3 \end{aligned} \tag{2.8}$$

Here **E** is the unit tensor. By relations (2.8) the rotation of material fibres when an elastic solid is bent consists of a sequence of two finite rotations: rotation by an angle $\omega(x_1, x_2)$ around the x_3 axis and rotation by an angle βx_3 around the x_1 axis.

The representation of the rotation tensor in terms of the Piola stress tensor is determined from the equation which expresses the symmetry of the tensor **S**

$$\mathbf{D}\cdot\mathbf{A}^T=\mathbf{A}\cdot\mathbf{D}^T \tag{2.9}$$

Using relations (1.7) and (2.8), Eq (2.9) can be reduced to the form

$$(D_{12}-D_{21})\cos\omega=(D_{11}+D_{22})\sin\omega \tag{2.10}$$

It is obvious that for pure bending of a rod, rotations around the rod axes are small, so that the inequality $|\omega|<\pi/2$ is necessarily satisfied. In view of the last condition. Eq. (2.10) has the unique solution

$$\cos\omega=\frac{p}{\Delta},\quad \sin\omega=\frac{pq}{|p|\Delta}$$
$$p=D_{11}+D_{22},\quad q=D_{12}-D_{21},\quad \Delta=\sqrt{p^2+q^2} \tag{2.11}$$

From (2.5), (2.8) and (2.11) we obtain expressions for the tensors **A** and **S** in terms of the Piola stress tensor for any isotropic material

$$\mathbf{A}=\frac{|p|}{\Delta}\left[(\mathbf{i}_1\mathbf{i}_1+\mathbf{i}_2\mathbf{e}_2)-\frac{q}{p}(\mathbf{i}_2\mathbf{i}_1-\mathbf{i}_1\mathbf{e}_2)\right]+\mathbf{i}_3\mathbf{e}_3 \tag{2.12}$$

$$\mathbf{S}=\frac{p}{|p|\Delta}[(D_{11}p+D_{12}q)\mathbf{i}_1\mathbf{i}_1+(D_{11}D_{21}+D_{12}D_{22})(\mathbf{i}_1\mathbf{i}_2+\mathbf{i}_2\mathbf{i}_1)+$$
$$+(D_{22}p-D_{21}q)\mathbf{i}_2\mathbf{i}_2]+D_{33}\mathbf{i}_3\mathbf{i}_3 \tag{2.13}$$

If, for a specified material, the function η in (2.6) is known, the representation of the deformation gradient in terms of the tensor **D** can be constructed using relations (2.7), (2.9) and (2.13).

We will obtain the function η for two common models of a non-linearly elastic solid: a compressible semi-linear material and a highly plastic incompressible Bartenev–Khazanovich material. The elastic potentials of these materials have the following respective forms [3]

$$W=\tfrac{1}{2}\lambda\,\mathrm{tr}^2(\mathbf{U}-\mathbf{E})+\mu\,\mathrm{tr}(\mathbf{U}-\mathbf{E})^2 \tag{2.14}$$

$$W=2\mu(\mathrm{tr}\mathbf{U}-3),\quad \det\mathbf{U}=1 \tag{2.15}$$

where λ and μ are constants. The constitutive relations, which follow from (2.14) and (2.15), can be written as follows:

$$\mathbf{S}=\lambda\mathbf{E}\,\mathrm{tr}(\mathbf{U}-\mathbf{E})+2\mu(\mathbf{U}-\mathbf{E}) \tag{2.16}$$

$$\mathbf{S}=2\mu\mathbf{E}-q_0\mathbf{U}^{-1} \tag{2.17}$$

Here q_0 is the pressure in the incompressible solid, which is not expressible in terms of the deformation. On the basis of (2.16) we obtain the representation $\mathbf{U}=\eta(\mathbf{S})$ for the semi-linear material

$$\mathbf{U}=\mathbf{E}+\frac{1}{2\mu}\left(\mathbf{S}-\frac{\nu}{1+\nu}\mathbf{E}\,\mathrm{tr}\,\mathbf{S}\right),\quad \nu=\frac{\lambda}{2(\lambda+\mu)} \tag{2.18}$$

A similar representation for a Bartenev–Khazanovich material can be derived from (2.17) by eliminating the pressure q_0 using the incompressibility condition and has the form

$$\mathbf{U}=[\det(\mathbf{S}-2\mu\mathbf{E})]^{1/3}(\mathbf{S}-2\mu\mathbf{E})^{-1} \tag{2.19}$$

3. THE STRESS FUNCTIONS AND THE VARIATIONAL FORMULATION OF THE BENDING PROBLEM

The equilibrium equations (1.8) are satisfied identically by the following substitution

$$D_{11} = \frac{\partial \Psi}{\partial x_2}, \quad D_{12} = \beta\Phi_2, \quad D_{21} = -\frac{\partial \Psi}{\partial x_1}, \quad D_{22} = -\beta\Phi_1, \quad D_{33} = \frac{\partial \Phi_2}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_2} \tag{3.1}$$

The functions $\Psi(x_1, x_2)$, $\Phi_1(x_1, x_2)$ $\Phi_2(x_1, x_2)$ will be called the stress functions. Introducing the vector $\Phi = \Phi_1 \mathbf{i}_1 + \Phi_2 \mathbf{i}_2$ and the nabla operator $\nabla = \mathbf{i}_1 \partial/\partial x_1 + \mathbf{i}_2 \partial/\partial x_2$ in the cross-section plane of the beam, we can write representation (3.1) in the following invariant (coordinate-free) form

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot (\nabla \Psi \mathbf{i}_1 + \beta \boldsymbol{\Phi} \mathbf{e}_2) + \nabla \cdot (\boldsymbol{\epsilon} \cdot \boldsymbol{\Phi}) \mathbf{i}_3 \mathbf{e}_3, \quad \boldsymbol{\epsilon} = -\mathbf{i}_3 \times \mathbf{E} \tag{3.2}$$

Using relations (3.2) it is easy to construct a representation of the Piola tensor in terms of the stress function in any curvilinear coordinates, introduced into the region σ. According to (3.2), boundary conditions (1.9) on the boundary $\partial\sigma$ of the cross-section of the rod can be written in terms of the stress functions as follows:

$$\mathbf{t} \cdot \boldsymbol{\Phi} = 0, \quad \partial \Psi / \partial s = 0 \tag{3.3}$$

where $\mathbf{t}$ is unit vector tangential to the curve $\partial\sigma$ and s is the actual length of the arc on $\partial\sigma$. Since the function Ψ is determined by the stressed state of the solid, apart from an additive constant, in the case of the simply connected region σ the second equation of (3.3) is equivalent to the condition $\Psi = 0$.

If the deformation gradient $\mathbf{C}$ is expressed in terms of the tensor $\mathbf{D}$, as described in Section 2, and the tensor $\mathbf{D}$ is expressed in terms of the stress function using (3.1), the two-dimensional boundary-value problem of the bending of a beam will consist of the three compatibility equations (2.3), written in terms of the stress functions Ψ, Φ_1, Φ_2, and the boundary conditions (3.3). This boundary-value problem allows of a variational formulation based on the principle of supplementary energy of the non-linear theory of elasticity [5, 7].

Consider the functional Π, defined on the set of stress functions, doubly differentiable in the region σ, which on $\partial\sigma$ satisfy the conditions $\mathbf{t} \cdot \Phi = \Psi = 0$,

$$\Pi = \iint_{\sigma} V[\mathbf{D}(\boldsymbol{\Phi}, \Psi)] d\sigma, \quad V(\mathbf{D}) = \operatorname{tr}[\mathbf{C}^T(\mathbf{D}) \cdot \mathbf{D}] - W(\mathbf{D}) \tag{3.4}$$

Here $V(\mathbf{D})$ is the specific supplementary energy, which is related to the specific potential energy $W(\mathbf{C})$ by a Legendre transformation $\mathbf{D}(\Phi, \Psi)$ – representation (3.2) of the Piola tensor in terms of the stress functions. Since $\mathrm{C} = \partial V/\partial \mathbf{D}$ it is easily verified that the condition for the functional Π to be stationary is equivalent to compatibility equations (2.3).

Using relations (2.14), (2.15), (2.18) and (2.19) we obtain the specific supplementary energy functions for a semi-linear material and a Bartenev–Khazanovich material respectively

$$V(\mathbf{S}) = \frac{1}{4\mu}\left(\operatorname{tr}\mathbf{S}^2 - \frac{\nu}{1+\nu}\operatorname{tr}^2\mathbf{S}\right) + \operatorname{tr}\mathbf{S} \quad \text{и} \quad V(\mathbf{S}) = 3[\det(\mathbf{S} - 2\mu\mathbf{E})]^{1/3} + 6\mu \tag{3.5}$$

The function $V(\mathbf{D})$ which occurs in functional (3.4) for the materials considered is obtained by substituting expression (2.13) into relations (3.5).

4. THE BENDING OF A BEAM OF RECTANGULAR CROSS-SECTION

In this case region σ is given by the inequalities

$$|x_1| \leqslant a, \quad |x_2| \leqslant b \tag{4.1}$$

where $2a$ and $2b$ are the width and height of the rectangle. In view of the symmetry of this region about the x_2 axis the oddness properties of the functions α, D_{12} and D_{21} and the evenness of the functions ρ, B_{11} and B_{33} with respect to the variable x_1, noted at the end of Section 1, hold. It follows from this, taking relations (1.2), (1.6) and (3.1) into account, that the stress functions Ψ and Φ_1 are even and the function Φ_2 is odd with respect to the variable x_1.

We will use Ritz's method to solve the variational problem of the stationarity of the functional Π. We will approximate the required stress functions Ψ Φ_1 Φ_2 by polynomials in x_1 and x_2, confining ourselves to terms up to the third power inclusive. Boundary conditions (1.9) on the side surface of the beam, taking relations (3.1) into account, can be written as follows:

$$\partial\Psi/\partial x_2\big|_{x_1=\pm a} = \Phi_2\big|_{x_1=\pm a} = 0, \quad \partial\Psi/\partial x_1\big|_{x_2=\pm b} = \Phi_1\big|_{x_2=\pm b} = 0 \tag{4.2}$$

By satisfying conditions (4.2) and the evenness properties, we obtain the following approximation of the required stress functions

$$\Phi_1 = A(x_2^2 - b^2), \quad \Phi_2 = Bx_1(x_1^2 - a^2), \quad \Psi = K \tag{4.3}$$

By relation (3.1) the representation of the Piola stress tensor, corresponding to approximation (4.3), has the form

$$\mathbf{D} = \beta Bx_1(x_1^2 - a^2)\mathbf{i}_1\mathbf{e}_2 - \beta A(x_2^2 - b^2)\mathbf{i}_2\mathbf{e}_2 + [2Ax_2 - B(3x_1^2 - a^2)]\mathbf{i}_3\mathbf{e}_3 \tag{4.4}$$

After substituting expressions (4.3) into the supplementary energy function Π, the latter becomes a function of the two variables A and B, while the condition for it to be stationary reduces to the equations

$$\partial\Pi/\partial A = 0, \quad \partial\Pi/\partial B = 0 \tag{4.5}$$

The system of non-linear equations (4.5) in the constants A and B were solved numerically for various of β. After determining the quantities A and B the bending moment M_1 is calculated from (1.11) and (4.4), taking into account the relation $\rho(x_1, x_2) = \beta^{-1}C_{33}(x_1, x_2)$, which follows from (1.2).

The last formula also enables us to obtain the curvature of the axis of the bent beam

$$1/\rho(0, 0) = \beta/C_{33}(0, 0) \tag{4.6}$$

Calculations show that the quantity $C_{33}(0,0)$ differs only slightly from unity both for a semi-linear material and for a Bartenev–Khazanovich material. Hence, by relation (4.6), the parameter β can be assumed, with a high degree of accuracy, to be the curvature of the axis of the deformed beam.

A graph of the bending moment against the curvature of the axis of the bent beam for a semi-linear material is shown in Fig. 1 (curve 1). Straight line 1 corresponds to classical linear bending theory, according to which

$$M_1 = \mu(1 + \nu)ab^3\beta/6 \tag{4.7}$$

In the calculations we assumed $\mu = 1$, $\nu = 0.3$, $a = 0.5$ and $b = 1.5$.

The deformation diagram of a beam of Bartenov–Khazanovich material represented by curve 2 for $\mu = 1$, $a = 0.5$ and $b = 1.5$. Note that, in the range of small deformations, the Bartenev–Khazanovich material obeys Hooke's

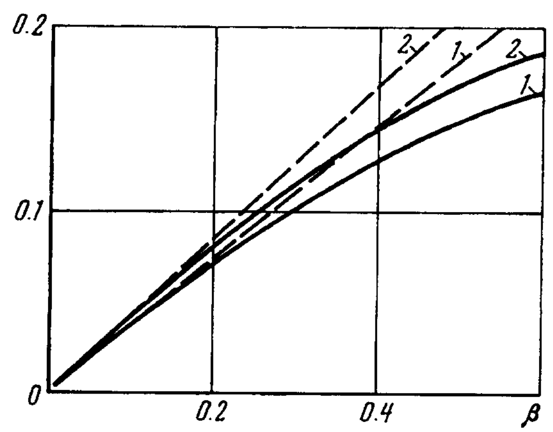

Fig. 1

law with a shear modulus μ and Poisson's ratio $\nu = 1/2$. Hence, straight line 2 in the figure is described by formula (4.7) with $\nu = 1/2$.

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